Volumes of Fano Manifolds

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This is supposed to be the note for a talk in the algebraic geometry student seminar.

The volume of a Fano manifold plays an important role in the study of Fano manifolds. One interesting question is to find upper bounds of the volumes for various classes of Fano manifolds. In this talk, I will consider Fano 3-folds first. By classification results of Fano 3-folds, we know that the volumes of Fano 3-folds are bounded by 64, which is the volume of the projective 3-space. I will also show how to compute explicitly upper bounds of the volumes for some particular classes of Fano 3-folds. This is not going to give any new upper bounds better than 64 for Fano 3-folds. However, similar questions become more interesting for higher dimensional Fano manifolds.

1 Introduction

A Fano manifold is a smooth projective variety such that the anti-canonical class is ample. The volume of a Fano manifold X of dimension n is defined to be $\mathrm{vol}(-K_X) = (-K_X)^n$.

Example 1.1. The volume of the projective space $\mathbb{P}^n$ is $(n+1)^n$.

Example 1.2. Let $X_{d_1 \cdots d_r} \subset \mathbb{P}^{n+r}$ be a complete intersection. Using adjunction, we have $(-K_X)^n = d_1 \cdots d_r(n+r+1-d_1-\cdots-d_r)^n$. In particular, a degree d hypersurface in $\mathbb{P}^{n+1}$ has volume $d(n+2-d)^n$.

Example 1.3. The degree of a del Pezzo surface is defined to be the volume $(-K_X)^2$. $\mathbb{P}^2$ is the del Pezzo surface with largest degree, which is 9.

2 Stability of Fano Manifolds

Let F be a prime divisor over X. Let $\pi : Y \to X$ be a proper birational morphism and F a prime divisor on Y. Let $A_X(F) = \mathrm{ord}_F(K_Y - \pi^* K_X) + 1$ be the log discrepancy of F, and

$$S_X(F) = \frac{1}{(-K_X)^n} \int_0^\infty \mathrm{vol}(-\pi^* K_X - xF)dx.$$

Definition 2.1 ([Fuj19, Li17]). A Fano manifold X is called K-semistable if $A_X(F) \geq S_X(F)$ for any prime divisor F over X.

Let H be an ample line bundle on X. The slope of a torsion-free sheaf $\mathcal{E}$ with respect to H is defined as

$$\mu_H(\mathcal{E}) = \frac{\deg_H \mathcal{E}}{\operatorname{rk} \mathcal{E}},$$

where $\deg_H \mathcal{E} = H^{n-1} \cdot c_1(\mathcal{E})$ and $\operatorname{rk} \mathcal{E}$ is the rank of $\mathcal{E}$.

Definition 2.2. A torsion-free sheaf $\mathcal{E}$ on X is called slope semistable with respect to H (or H-semistable) if for any subsheaf $0 \neq \mathcal{F} \subset \mathcal{E}$, we have

$$\mu_H(\mathcal{F}) \leq \mu_H(\mathcal{E})$$

We have the following Bogomolov-Gieseker inequality:

Theorem 2.3 ([Miy87]). Let X be a normal projective variety and $\mathcal{E}$ a rank r H-semistable sheaf on X. Then

$$H^{n-2} \cdot \Delta(\mathcal{E}) \geq 0,$$

where $\Delta(\mathcal{E}) = 2rc_2(\mathcal{E}) - (r-1)c_1(\mathcal{E})^2$ is the Bogomolov discriminant of $\mathcal{E}$.

Definition 2.4. A Fano manifold is called Bogomolov semistable if the tangent bundle T_X is $(-K_X)$-semistable.

Theorem 2.5 ([Li18]). A K-semistable Fano manifold is Bogomolov semistable.

Remark. A Kähler-Einstein metric is an Hermitian-Einstein metric on the tangent bundle. We know that the existence of a Kähler-Einstein metric is equivalent to K-polystability of a Fano manifold, whereas the existence of an Hermitian-Einstein metric on the tangent bundle is equivalent to $(-K_X)$-polystability of the tangent bundle. Therefore it follows immediately that K-polystability of X implies that T_X is $(-K_X)$-polystable.

3 Volume of Fano 3-fold

For K-semistable Fano manifold, the volume is bounded by the volume of the projective space of the same dimension due to Fujita.

Theorem 3.1 ([Fuj18]). Let X be a K-semistable Fano manifold. Then $(-K_X) \leq (n+1)^n$.

Proof. Blow up any smooth point in X and let E be the exceptional divisor. Using the inequality $A_X(E) \geq S_X(E)$, and

$$\operatorname{vol}(-\pi^* K_X - xE) \geq (-K_X)^n - x^n,$$

we can get the result. □

In particular, for Fano 3-folds, we have $(-K_X)^3 \leq 64$.

For Bogomolov semistable Fano 3-folds, we can also bound the volume.

Proposition 3.2. Let X be a Bogomolov semistable Fano 3-fold. Then $(-K_X)^3 \leq 72$.

Proof. Using Bogomolov-Gieseker inequality, we have

$$(-K_X)(6c_2(X) - 2c_1(X)^2) \geq 0$$

Note that $-K_X$ can be viewed as a class representing $c_1(X)$, so we have

$$(-K_X)^3 \leq 3c_1(X)c_2(X).$$

Apply Hirzebruch–Riemann–Roch to $\mathcal{O}_X$, we have

$$\chi(\mathcal{O}_X) = \int_X ch(\mathcal{O}_X) \cdot \mathrm{td}(X) = \mathrm{td}_3(X).$$

Using Kodaira Vanishing, we have $H^i(X, OS_X) = 0$ for all $i > 0$. Also $\mathrm{td}_3(X) = c_1(X)c_2(X)/24$. Therefore we know that $c_1(X)c_2(X) = 24$ and $(-K_X)^3 \leq 72$. $\square$

However, the above results are covered by the classification results of Fano 3-folds. The whole table of Fano 3-folds can be found in the appendix of [IPP$^+$10]. And therefore we have the volume of all Fano 3-folds.

Theorem 3.3. Let X be a Fano 3-fold. Then $(-K_X)^3 \leq 64$.

4 Volumes of Fano Manifolds of Higher Dimension

For Fano manifolds of dimension $n \geq 4$, there are examples with volume $(-K_X)^n > (n+1)^n$.

Example 4.1. Consider the projective bundle $X = \mathbb{P}_{\mathbb{P}^{n-1}}(\mathcal{O}_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(1-n))$. Using the relative Euler sequence of projective bundle $\pi : \ X = \mathbb{P}_Y(\mathcal{E}) \to Y$

$$0 \to \Omega_{X/Y} \to \mathcal{O}_X(-1) \otimes \pi^*\mathcal{E} \to \mathcal{O}_X \to 0,$$

we have that the volume $(-K_X)^n = \frac{(2n-1)^n - 1}{n-1} > (n+1)^n$ when $n \geq 4$.

Note that the example above has Picard rank 2. For Picard rank 1 Fano manifolds, we have the following conjecture.

Conjecture. Let X be a Fano manifold of Picard rank 1. Then $(-K_X)^n \leq (n+1)^n$.

The conjecture is true for $n = 4$ due to [Hwa03].

For K-semistable Fano manifolds, we still have that $(-K_X)^n \leq (n+1)^n$ by Fujita's result. However, for Bogomolov semistable Fano manifolds, by the Bogomolov-Gieseker inequality, we only have

$$(-K_X)^n \leq \frac{2n}{n-1} c_1(X)^{n-2} c_2(X).$$

It is still not clear how we can bound the chern class on the right-hand side and hence the volume of these Fano manifolds.

Also note that the example above is not Bogomolov semistable and hence also not K-semistable. It would be interesting to find exmaples of Bogomolov semistable but not K-semistable Fano manifolds, and whether there are Bogomolov semistable Fano manifolds with $(-K_X)^n > (n+1)^n$.

Another intresting question is about the relation between slope stability and Bogomolov stability of Fano manifolds.

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